Fourier Series

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Fourier Series

Periodic Functions

The Mathematic Formulation

- Any function that satisfies

$$f(t) = f(t + mT) \quad m \in \mathbb{Z}$$

where T is a constant and is called the *period* of the function.

Example:

$$f(t) = \cos\frac{t}{3} + \cos\frac{t}{4}$$ Find its period.

$$f(t) = f(t+T) \Rightarrow \cos\frac{t}{3} + \cos\frac{t}{4} = \cos\frac{1}{3}(t+T) + \cos\frac{1}{4}(t+T)$$

Fact: $\cos\theta = \cos(\theta + 2m\pi)$

$$\frac{T}{3} = 2m\pi \qquad \frac{T}{4} = 2n\pi$$

$$\Rightarrow \quad T = 6m\pi \qquad T = 8n\pi$$

$$\Rightarrow \quad T = 24\pi$$ smallest T

Example:

$f(t) = \cos\omega_1 t + \cos\omega_2 t$ Find its period.

$f(t) = f(t+T)$ ⇒ $\cos\omega_1 t + \cos\omega_2 t = \cos\omega_1(t+T) + \cos\omega_2(t+T)$

$\omega_1 T = 2m\pi$

$\omega_2 T = 2n\pi$

⇒ $\dfrac{\omega_1}{\omega_2} = \dfrac{m}{n}$ ⇒ $\dfrac{\omega_1}{\omega_2}$ must be a rational number

Example:

$$f(t) = \cos 10t + \cos(10+\pi)t$$

Is this function a periodic one?

$$\frac{\omega_1}{\omega_2} = \frac{10}{10+\pi}$$

not a rational number

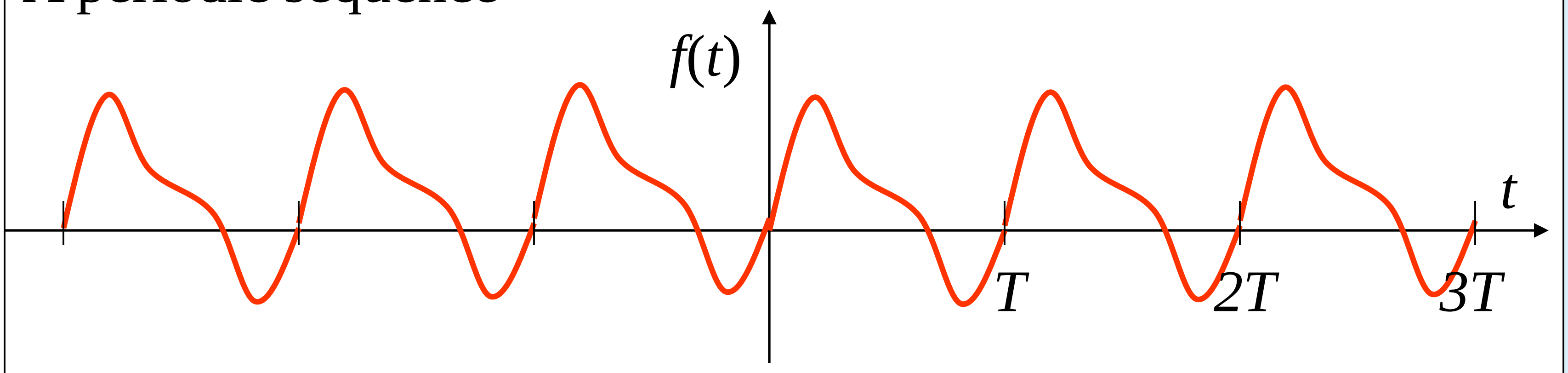
Fourier Series

Fourier Series

Introduction

- Decompose a periodic input signal into *primitive periodic components*.

A periodic sequence



Synthesis

$$f(t) = \underbrace{\frac{a_0}{2}}_{\text{DC Part}} + \underbrace{\sum_{n=1}^{\infty} a_n \cos \frac{2\pi n t}{T}}_{\text{Even Part}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin \frac{2\pi n t}{T}}_{\text{Odd Part}}$$

T is a period of all the above signals

Let $\omega_0 = 2\pi/T$.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

Orthogonal Functions

- Call a set of functions $\{\phi_k\}$ *orthogonal* on an interval $a < t < b$ if it satisfies

$$\int_a^b \varphi_m(t)\varphi_n^*(t)dt = \begin{cases} 0 & m \neq n \\ N & m = n \end{cases} \quad or$$

$$\langle \varphi_m(t), \varphi_n(t) \rangle = N\delta_{m,n}$$

Orthogonal set of Sinusoidal Functions

Define $\omega_0 = 2\pi/T$.

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)dt = 0, \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)dt = 0, \qquad \forall m$$

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases} = \frac{T}{2}\delta_{m,n}$$

We now prove this one

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)dt = \frac{T}{2}\delta_{m,n}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)dt = 0, \quad \text{for all } m \text{ and } n$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

Proof

$$\int_{-T/2}^{T/2}\cos(m\omega_0 t)\cos(n\omega_0 t)dt \qquad m \neq n$$

$$= \frac{1}{2}\int_{-T/2}^{T/2}\cos[(m+n)\omega_0 t]dt + \frac{1}{2}\int_{-T/2}^{T/2}\cos[(m-n)\omega_0 t]dt$$

$$= \frac{1}{2}\frac{1}{(m+n)\omega_0}\sin[(m+n)\omega_0 t]\Big|_{-T/2}^{T/2} + \frac{1}{2}\frac{1}{(m-n)\omega_0}\sin[(m-n)\omega_0 t]\Big|_{-T/2}^{T/2}$$

$$= \frac{1}{2}\frac{1}{(m+n)\omega_0}\underbrace{2\sin[(m+n)\pi]}_{0} + \frac{1}{2}\frac{1}{(m-n)\omega_0}\underbrace{2\sin[(m-n)\pi]}_{0}$$

$$= 0$$

Proof

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

$$\cos^2\alpha = \frac{1}{2}[1+\cos 2\alpha]$$

$$\int_{-T/2}^{T/2}\cos(m\omega_0 t)\cos(n\omega_0 t)dt \qquad m = n$$

$$= \int_{-T/2}^{T/2}\cos^2(m\omega_0 t)dt \quad = \frac{1}{2}\int_{-T/2}^{T/2}[1+\cos 2m\omega_0 t]dt$$

$$= \frac{1}{2}t\Big|_{-T/2}^{T/2} + \underbrace{\frac{1}{4m\omega_0}\sin 2m\omega_0 t]\Big|_{-T/2}^{T/2}}_{0}$$

$$= \frac{T}{2}$$

$$\int_{-T/2}^{T/2}\cos(m\omega_0 t)\cos(n\omega_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

Orthogonal set of Sinusoidal Functions

Define $\omega_0 = 2\pi / T$.

$\int_{-T/2}^{T/2} c \ldots \quad m \neq 0$

$\int_{-T/2}^{T/2} c \ldots$

$$\begin{Bmatrix} 1, \\ \cos\omega_0 t, \cos 2\omega_0 t, \cos 3\omega_0 t, \cdots \\ \sin\omega_0 t, \sin 2\omega_0 t, \sin 3\omega_0 t, \cdots \end{Bmatrix}$$

$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)dt = \{$

an orthogonal set.

$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)dt = 0, \quad$ for all m and n

Decomposition

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$$a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)dt$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t dt \qquad n = 1,2,\cdots$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t dt \qquad n = 1,2,\cdots$$

Convergence

Let $f(t), T$ picewise continuous periodic function

$$f_N(t) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(n\omega_0 t) + \sum_{n=1}^{N} b_n \sin(n\omega_0 t)$$

$$1 - \forall t \in \mathbb{R} \quad \lim_{N \to \infty} f_N(t) = \frac{1}{2}\left[f(t^+) - f(t^-) \right]$$

If $f(t)$ continuous at t_0 the limit is $f(t_0)$

2- if $f(t)$ continuous in $\mathbb{R}$ and $f^{'}(t)$ picewise continuous function within $(\mathrm{a,b})$ interval, $f_N(t)$ CVU to $f(t)$

(Suite)

3- if $f''(t)$ exists except on a finite number points on $\forall$ closed interval, then in a point t_0 where $f''(t_0)$ exists then

$$\lim_{N\to\infty} f_N'(t) = f'(t)$$

Convergence and existence

1- absolutely integrable over any period

$$\int_{\langle T_0 \rangle} \left| f(t) \right| dt \langle \infty$$

2- $f(t)$ has a finite number of extremum within any finite interval of t

3- $f(t)$ has a finite number of discontinuities within any interval of t and each of these discontinuities is finite

….sufficient but not necessary conditions

Proof

Use the following facts:

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)dt = 0, \qquad m \neq 0$$

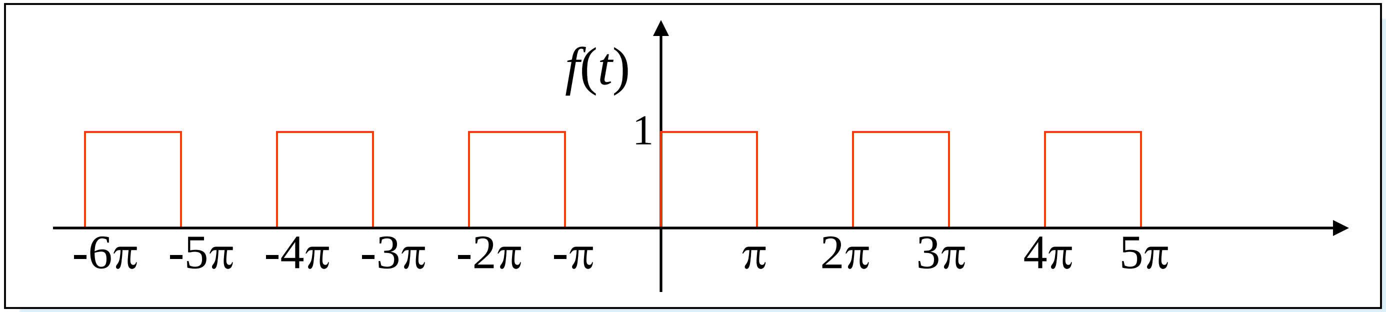
$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)dt = 0, \qquad m \neq 0$$

$$\int_{-T/2}^{T/2} \cos(m\omega_0 t)\cos(n\omega_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\sin(n\omega_0 t)dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

$$\int_{-T/2}^{T/2} \sin(m\omega_0 t)\cos(n\omega_0 t)dt = 0, \quad \text{for all } m \text{ and } n$$

Example (Square Wave)



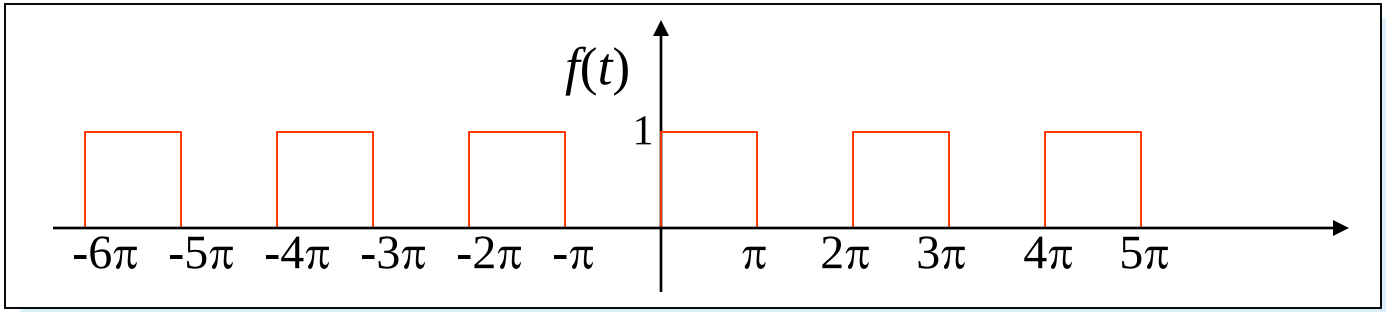
$$a_0 = \frac{2}{2\pi}\int_0^{\pi} 1dt = 1 \quad : \quad T = 2\pi$$

$$a_n = \frac{2}{2\pi}\int_0^{\pi} \cos nt dt = \frac{1}{n\pi}\sin nt\Big|_0^{\pi} = 0 \qquad n = 1,2,\cdots$$

$$b_n = \frac{2}{2\pi}\int_0^{\pi} \sin nt dt = -\frac{1}{n\pi}\cos nt\Big|_0^{\pi} = -\frac{1}{n\pi}(\cos n\pi - 1) = \begin{cases} 2/n\pi & n = 1,3,5,\cdots \\ 0 & n = 2,4,6,\cdots \end{cases}$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right)$$

Example (Square Wave)



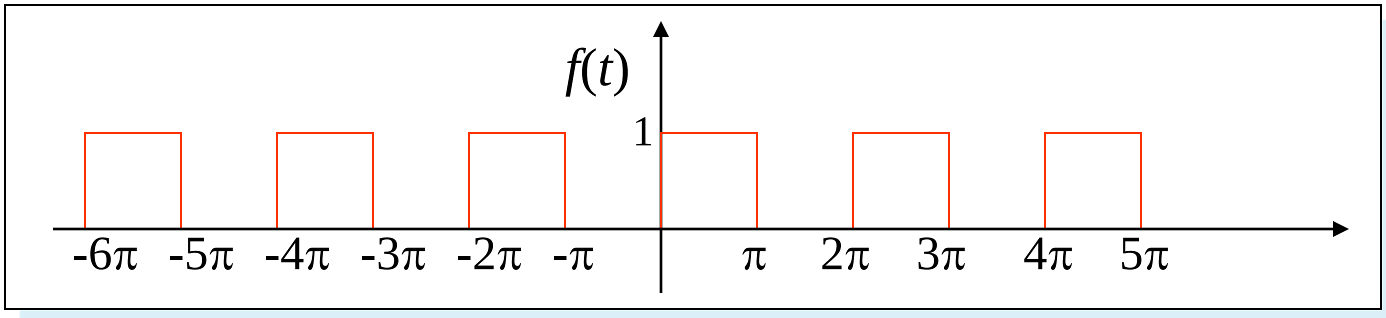
$$a_0 = \frac{2}{2\pi}\int_0^{\pi} 1\,dt = 1$$

$$a_n = \frac{2}{2\pi}\int_0^{\pi} \cos nt\,dt = \frac{1}{n\pi}\sin nt\Big|_0^{\pi} = 0 \qquad n = 1,2,\cdots$$

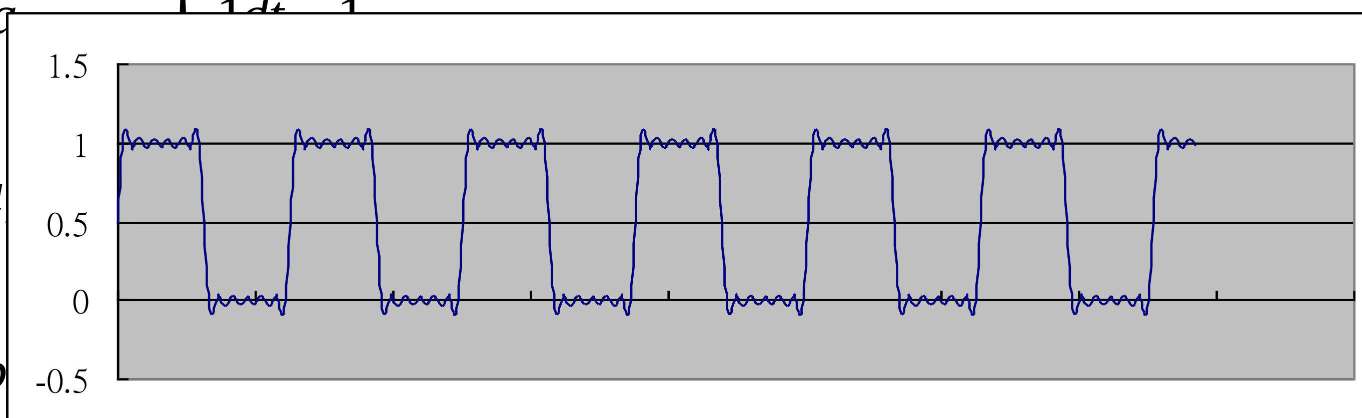
$$b_n = \frac{1}{2\pi}\int_0^{\pi} \sin nt\,dt = -\frac{1}{n\pi}\cos nt\Big|_0^{\pi} = -\frac{1}{n\pi}(\cos n\pi - 1) = \begin{cases} 2/n\pi & n = 1,3,5,\cdots \\ 0 & n = 2,4,6,\cdots \end{cases}$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right)$$

Example (Square Wave)



$\frac{2}{\,}\int^{\pi} 1dt = 1$



$n = 1,3,5,\cdots$

$n = 2,4,6,\cdots$

Harmonics

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi nt}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi nt}{T}$$

$$f(t) = \underbrace{\frac{a_0}{2}}_{\text{DC Part}} + \underbrace{\sum_{n=1}^{\infty} a_n \cos(n\omega_0 t)}_{\text{Even Part}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)}_{\text{Odd Part}}$$

DC Part — Even Part — Odd Part

T is a period of all the above signals

Harmonics

Define $\omega_0 = 2\pi f_0 = \dfrac{2\pi}{T}$, called the *fundamental angular frequency*.

Define $\omega_n = n\omega_0$, called the *n-th harmonic* of the periodic function.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$\Rightarrow$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \sin \omega_n t$$

Harmonics

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\omega_n t + \sum_{n=1}^{\infty} b_n \sin\omega_n t$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos\omega_n t + b_n \sin\omega_n t)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos\omega_n t + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin\omega_n t \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left(\cos\theta_n \cos\omega_n t + \sin\theta_n \sin\omega_n t \right)$$

$$= C_0 + \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \theta_n)$$

Amplitudes and Phase Angles

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \theta_n)$$

harmonic amplitude

phase angle

$$C_0 = \frac{a_0}{2}$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

$$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

Fourier Series

Complex Form of the Fourier Series

Complex Exponentials

$$e^{jn\omega_0 t} = \cos n\omega_0 t + j\sin n\omega_0 t$$

$$e^{-jn\omega_0 t} = \cos n\omega_0 t - j\sin n\omega_0 t$$

$$\cos n\omega_0 t = \frac{1}{2}\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right)$$

$$\sin n\omega_0 t = \frac{1}{2j}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right) = -\frac{j}{2}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

Complex Form of the Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$= \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty} a_n \left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right) - \frac{j}{2}\sum_{n=1}^{\infty} b_n \left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{2}(a_n - jb_n)e^{jn\omega_0 t} + \frac{1}{2}(a_n + jb_n)e^{-jn\omega_0 t}\right]$$

$$= c_0 + \sum_{n=1}^{\infty} \left[c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t}\right]$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

Complex Form of the Fourier Series

$$f(t) = c_0 + \sum_{n=1}^{\infty} \left[c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t} \right]$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} c_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

Complex Form of the Fourier Series

$$c_0 = \frac{a_0}{2} = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$= \frac{1}{T}\left[\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t dt - j\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t dt\right]$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)(\cos n\omega_0 t - j\sin n\omega_0 t)dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n) = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{jn\omega_0 t}dt$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

Complex Form of the Fourier Series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t}\,dt$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

If $f(t)$ is real,

$$c_n = |c_n| e^{j\phi_n}, \quad c_{-n} = c_n^* = |c_n| e^{-j\phi_n}$$

$$|c_n| = |c_{-n}| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

$$n = \pm 1, \pm 2, \pm 3, \cdots$$

$$c_{-n} = c_n^*$$

hermitian symmetry

$$\phi_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right)$$

$$c_0 = \frac{1}{2} a_0$$

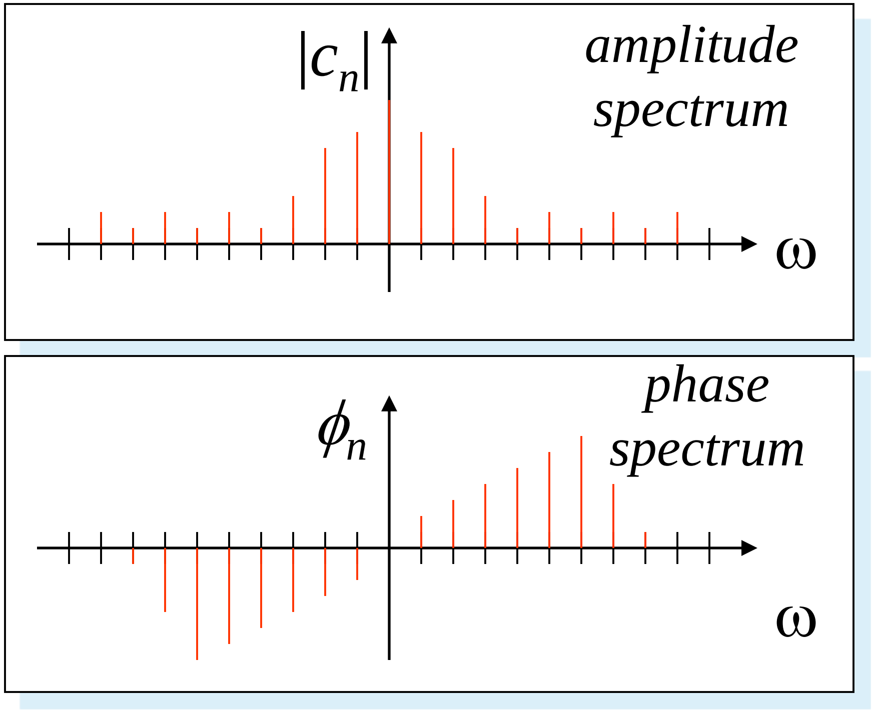
Complex Frequency Spectra

$$c_n = |c_n| e^{j\phi_n}, \quad c_{-n} = c_n^* = |c_n| e^{-j\phi_n}$$

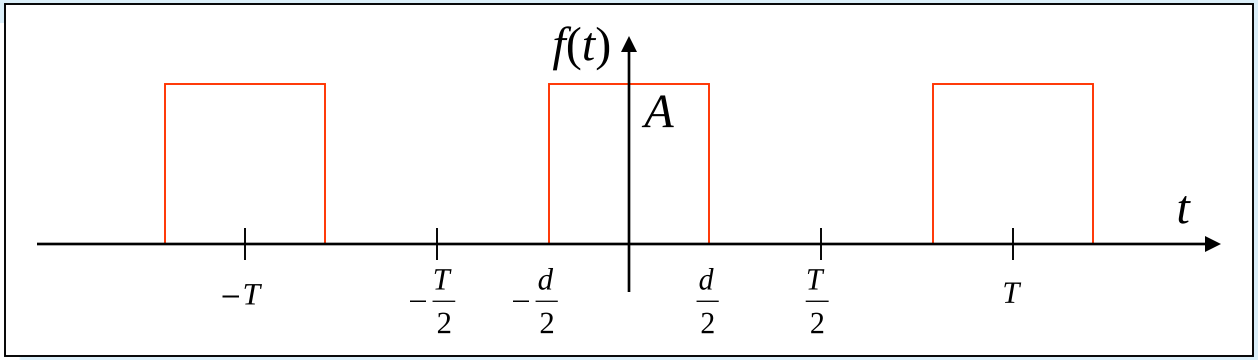
$$|c_n| = |c_{-n}| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

$$c_0 = \frac{1}{2} a_0$$

$$\phi_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right) \qquad n = \pm 1, \pm 2, \pm 3, \cdots$$



Example



$$c_n = \frac{A}{T}\int_{-d/2}^{d/2} e^{-jn\omega_0 t}\,dt$$

$$= \frac{A}{T}\frac{1}{-jn\omega_0} e^{-jn\omega_0 t}\Bigg|_{-d/2}^{d/2}$$

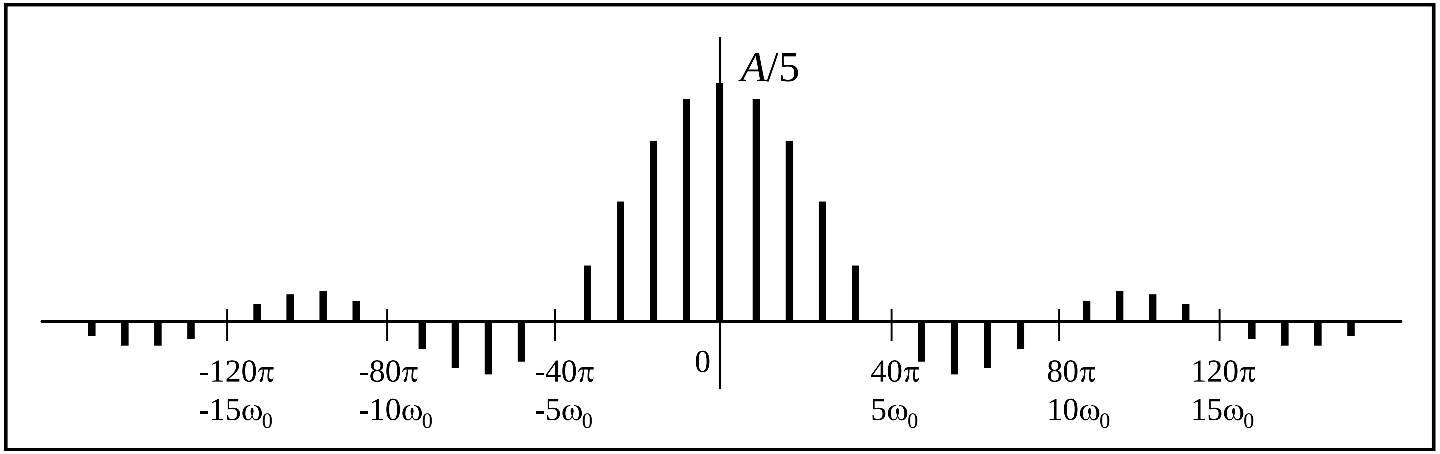
$$= \frac{A}{T}\left(\frac{1}{-jn\omega_0} e^{-jn\omega_0 d/2} - \frac{1}{-jn\omega_0} e^{jn\omega_0 d/2}\right)$$

$$= \frac{A}{T}\frac{1}{-jn\omega_0}(-2j\sin n\omega_0 d/2)$$

$$= \frac{A}{T}\frac{1}{\frac{1}{2}n\omega_0}\sin n\omega_0 d/2$$

$$= \frac{Ad}{T}\frac{\sin\left(\frac{n\pi d}{T}\right)}{\left(\frac{n\pi d}{T}\right)} = \frac{Ad}{T}\operatorname{sin}c\,\frac{n\pi d}{T}$$

Example

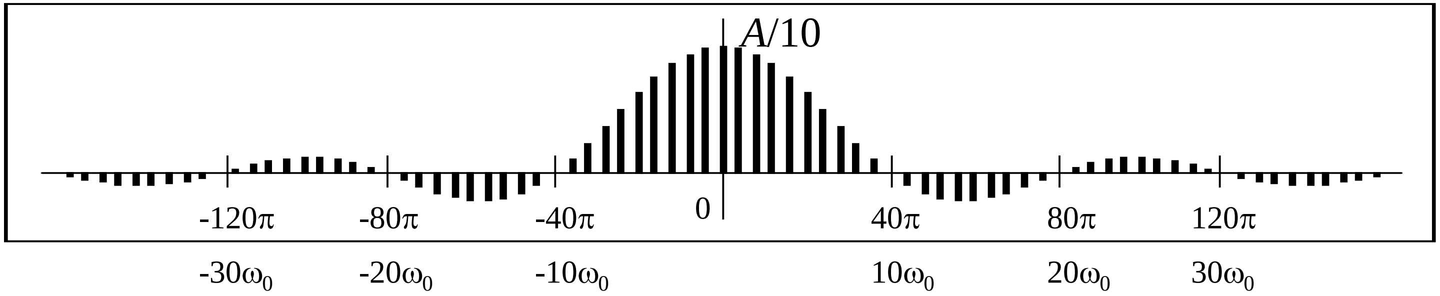


$$c_n = \frac{Ad}{T} \frac{\sin\left(\frac{n\pi d}{T}\right)}{\left(\frac{n\pi d}{T}\right)}$$

$$d = \frac{1}{20},\ T = \frac{1}{4},\ \frac{d}{\mathrm{T}} = \frac{1}{5}$$

$$\omega_0 = \frac{2\pi}{T} = 8\pi$$

Example

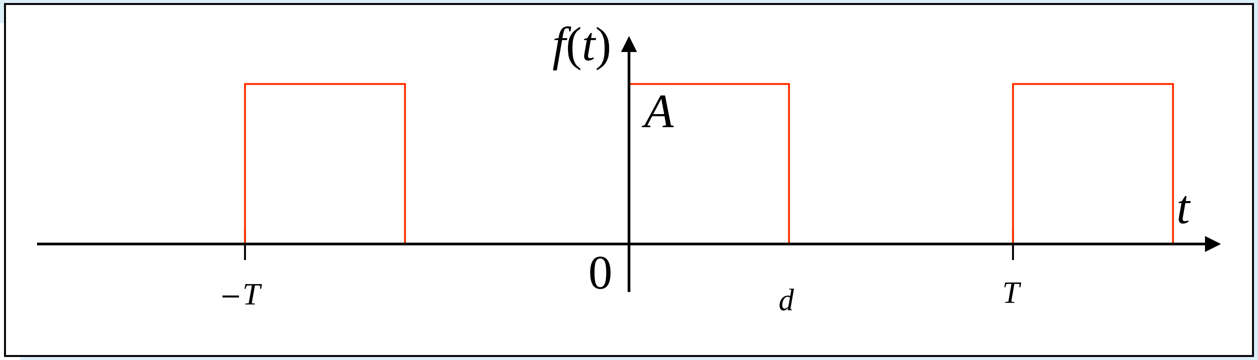


$$c_n = \frac{Ad}{T} \frac{\sin\left(\frac{n\pi d}{T}\right)}{\left(\frac{n\pi d}{T}\right)}$$

$$d = \frac{1}{20}, \quad \boxed{T = \frac{1}{2}}, \quad \frac{d}{\mathrm{T}} = \frac{1}{5}$$

$$\omega_0 = \frac{2\pi}{T} = 4\pi$$

Example



$$c_n = \frac{A}{T}\int_0^d e^{-jn\omega_0 t}\,dt$$

$$= \frac{A}{T}\frac{1}{-jn\omega_0} e^{-jn\omega_0 t}\Big|_0^d$$

$$= \frac{A}{T}\left(\frac{1}{-jn\omega_0} e^{-jn\omega_0 d} - \frac{1}{-jn\omega_0}\right)$$

$$= \frac{A}{T}\frac{1}{jn\omega_0}(1 - e^{-jn\omega_0 d})$$

$$= \frac{A}{T}\frac{1}{jn\omega_0} e^{-jn\omega_0 d/2}(e^{jn\omega_0 d/2} - e^{-jn\omega_0 d/2})$$

$$= \frac{Ad}{T}\frac{\sin\left(\frac{n\pi d}{T}\right)}{\left(\frac{n\pi d}{T}\right)} e^{-jn\omega_0 d/2}$$

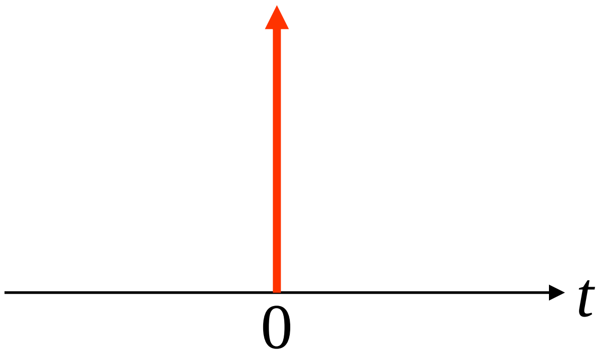
Fourier Series

Impulse Train

Dirac Delta "Function"

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$ and $$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)\,dt = 1 \quad \forall \varepsilon \rangle 0$$



Also called *unit impulse function*.

Generalized function

- But an ordinary function which is everywhere 0 except at a single point must have integral 0 (in the Riemann integral sense).
  - Thus $\delta(t)$ cannot be an ordinary function and mathematically it is defined by

$$\int_{-\infty}^{\infty} \delta(t)\varphi(t)dt = \varphi(0)$$

Where $\varphi(t)$ is an regular function continuous at $t = 0$

Generalized function (suite)

- An alternative definition of $\delta(t)$ is given by

$$\int_a^b \delta(t)\varphi(t)dt = \begin{cases} \varphi(0) & a\langle 0\langle b \\ 0 & a\langle b\langle 0 \quad or \quad 0\langle a\langle b \\ undefined & a=0 \quad or \quad b=0 \end{cases}$$

- It is why, $\delta(t)$ is often called a generalized function and $\varphi(t)$ a testing function $\in \mathcal{S}$

- Also

$\delta(t-t_0)$ *is defined by*

$$\int_{-\infty}^{\infty}\delta(t-t_0)\varphi(t)dt=\varphi(t_0)$$

Some properties of $\delta(t)$

Some properties

$$\delta(at) = \frac{1}{|a|}\delta(t) \qquad \delta(-t) = \delta(t)$$

$$x(t)\delta(t) = x(0)\delta(t) \qquad x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t) = x(t)*\delta(t)$$

Generalized derivatives

$$\int_{-\infty}^{\infty} g^{(n)}(t)\varphi(t)dt = (-1)^n \int_{-\infty}^{\infty} g(t)\varphi^{(n)}(t)dt$$

$and\ (because\ \varphi(t)\ vanished\ out\,side\,some\ fixed\ \text{int}\,erval)$

$$\int_{-\infty}^{\infty} \delta'(t)\varphi(t)dt = -\varphi'(0)$$

$$example \quad \delta(t) = \frac{du(t)}{dt}$$

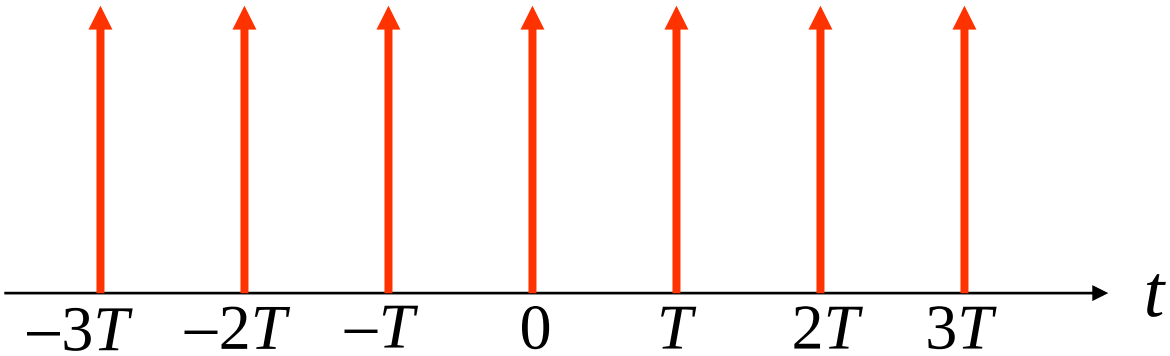
Property

$$\int_{-\infty}^{\infty} \delta(t)\phi(t)dt = \phi(0)$$

ϕ(t): Test Function

$$\int_{-\infty}^{\infty} \delta(t)\phi(t)dt = \int_{-\infty}^{\infty} \delta(t)\phi(0)dt = \phi(0)\int_{-\infty}^{\infty} \delta(t)dt = \phi(0)$$

Impulse Train



$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Fourier Series of the Impulse Train

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} \delta_T(t)dt = \frac{2}{T}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \delta_T(t)\cos(n\omega_0 t)dt = \frac{2}{T}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} \delta_T(t)\sin(n\omega_0 t)dt = 0$$

$$\delta_T(t) = \frac{1}{T} + \frac{2}{T}\sum_{n=-\infty}^{\infty} \cos n\omega_0 t$$

Complex Form
Fourier Series of the Impulse Train

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$c_0 = \frac{a_0}{2} = \frac{1}{T}\int_{-T/2}^{T/2} \delta_T(t)dt = \frac{1}{T}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta_T(t)e^{-jn\omega_0 t}dt = \frac{1}{T}$$

$$\delta_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

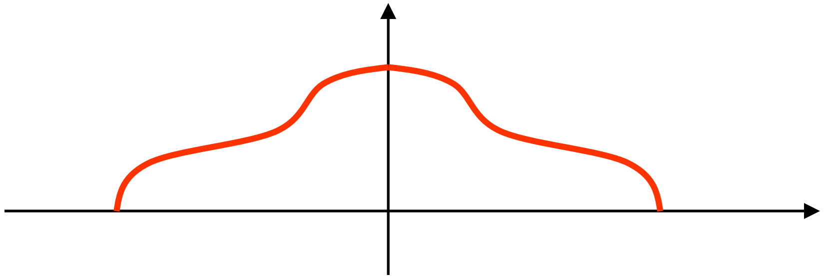
Fourier Series

Analysis of Periodic Waveforms

Waveform Symmetry

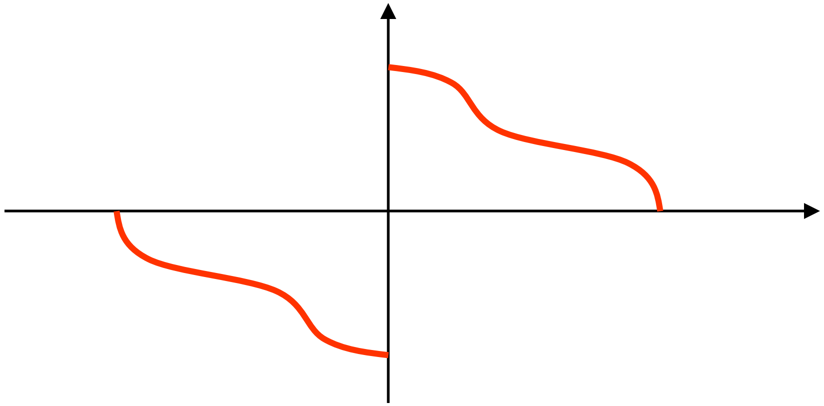
- Even Functions

$$f(t) = f(-t)$$



- Odd Functions

$$f(t) = -f(-t)$$



Decomposition

- Any function $f(t)$ can be expressed as the sum of an even function $f_e(t)$ and an odd function $f_o(t)$.

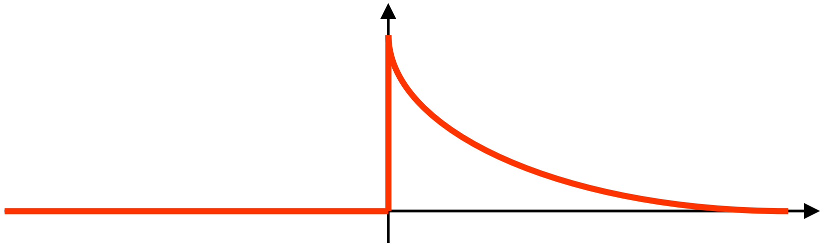
$$f(t) = f_e(t) + f_o(t)$$

$$f_e(t) = \tfrac{1}{2}[f(t) + f(-t)]$$ Even Part

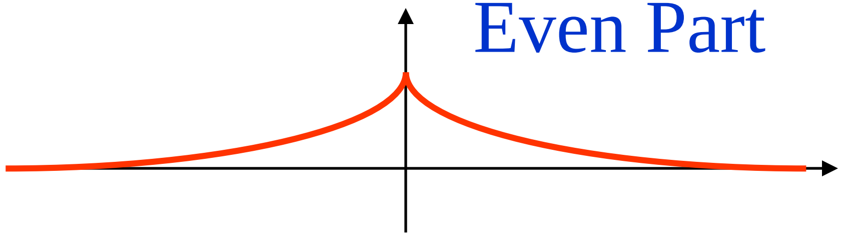
$$f_o(t) = \tfrac{1}{2}[f(t) - f(-t)]$$ Odd Part

Example

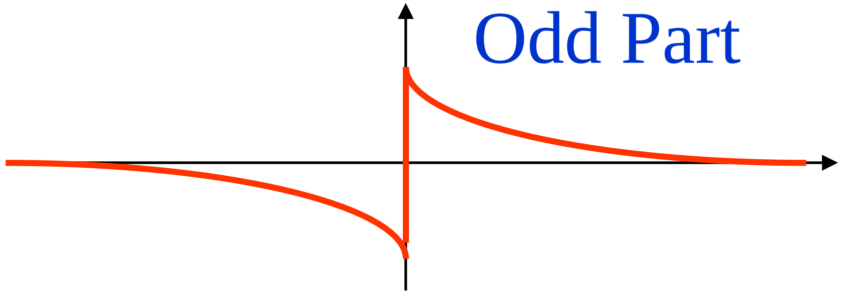
$$f(t)=\begin{cases} e^{-t} & t>0 \\ 0 & t<0 \end{cases}$$



$$f_e(t)=\begin{cases} \frac{1}{2}e^{-t} & t>0 \\ \frac{1}{2}e^{t} & t<0 \end{cases}$$

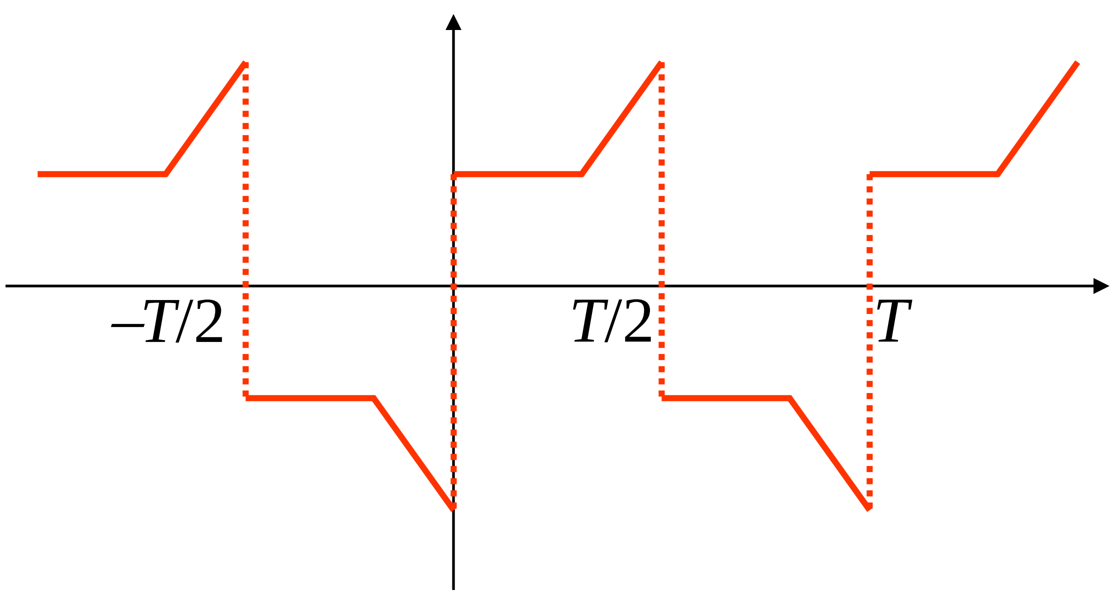


$$f_o(t)=\begin{cases} \frac{1}{2}e^{-t} & t>0 \\ -\frac{1}{2}e^{t} & t<0 \end{cases}$$



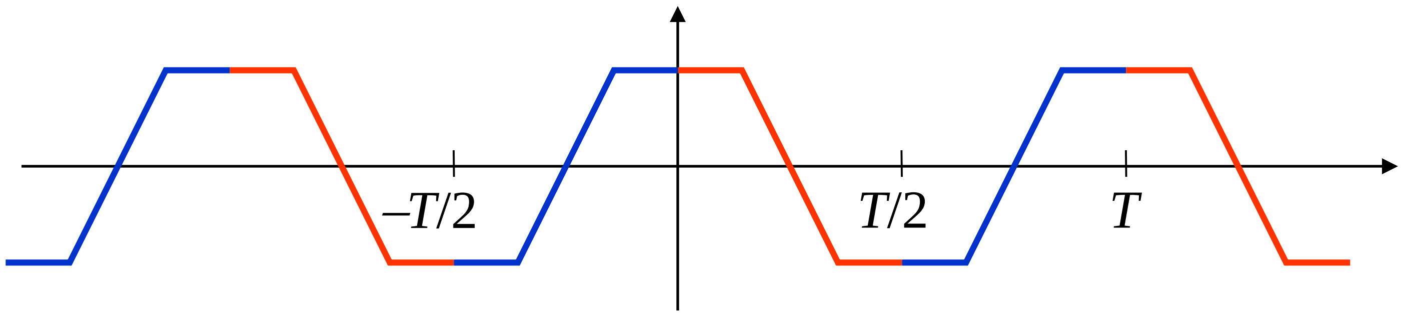
Half-Wave Symmetry

$$f(t) = f(t+T) \quad \text{and} \quad f(t) = -f\left(t + T/2\right)$$

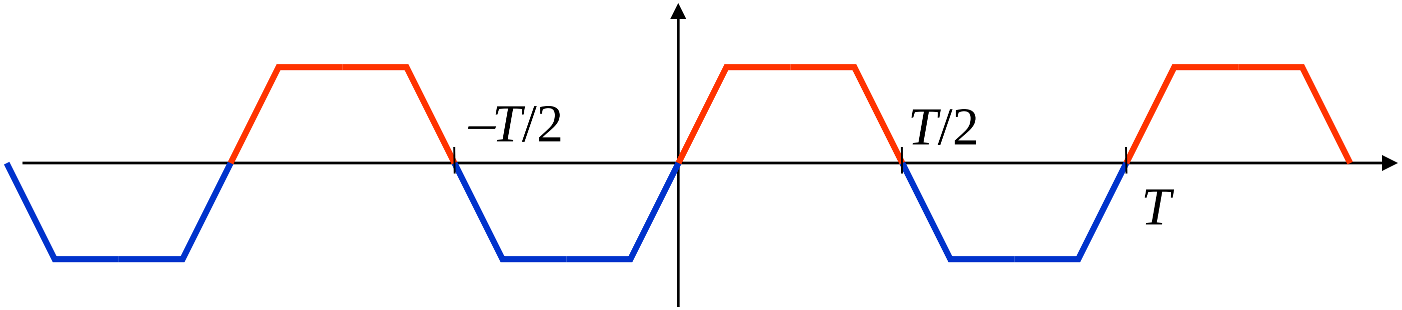


Quarter-Wave Symmetry

Even Quarter-Wave Symmetry

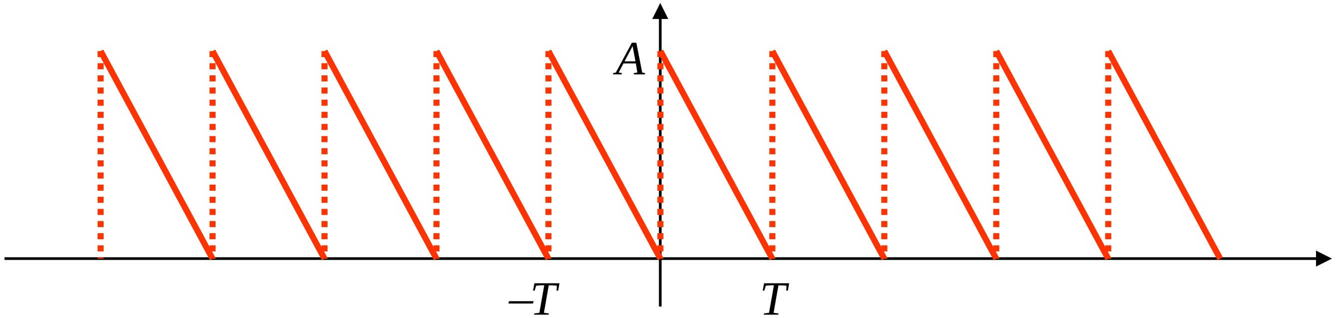


Odd Quarter-Wave Symmetry

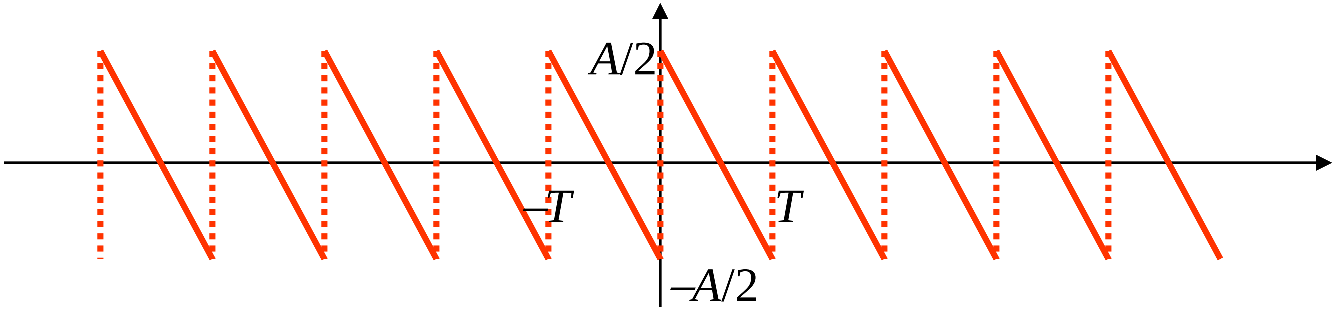


Hidden Symmetry

- The following is a asymmetry periodic function:



- Adding a constant to get symmetry property.

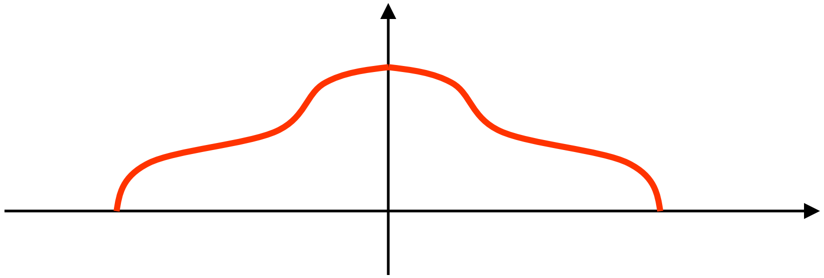


Fourier Coefficients of Symmetrical Waveforms

- The use of symmetry properties simplifies the calculation of Fourier coefficients.
  - Even Functions
  - Odd Functions
  - Half-Wave
  - Even Quarter-Wave
  - Odd Quarter-Wave
  - Hidden

Fourier Coefficients of Even Functions

$$f(t) = f(-t)$$

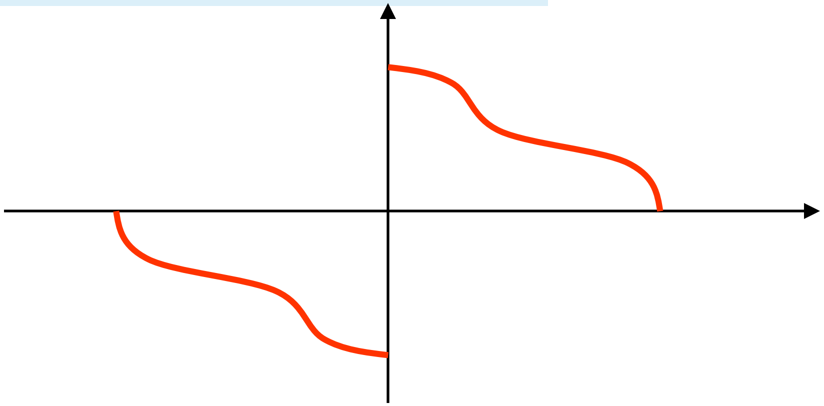


$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega_0 t)dt$$

Fourier Coefficients of Even Functions

$$f(t) = -f(-t)$$

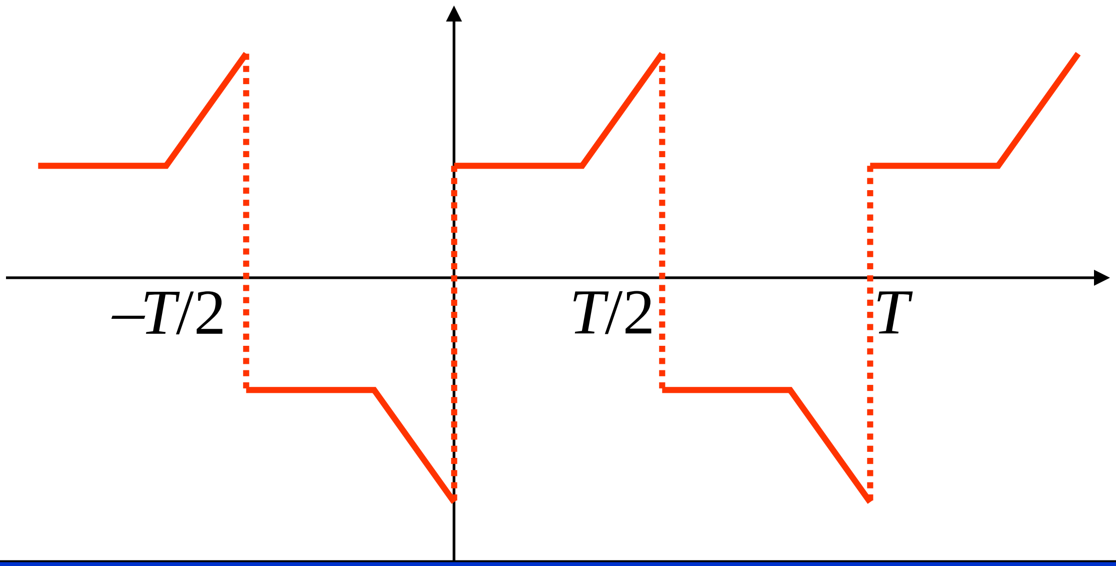


$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\omega_0 t)\,dt$$

Fourier Coefficients for Half-Wave Symmetry

$$f(t) = f(t+T) \quad \text{and} \quad f(t) = -f\left(t + T/2\right)$$



The Fourier series contains only odd harmonics.

Fourier Coefficients for Half-Wave Symmetry

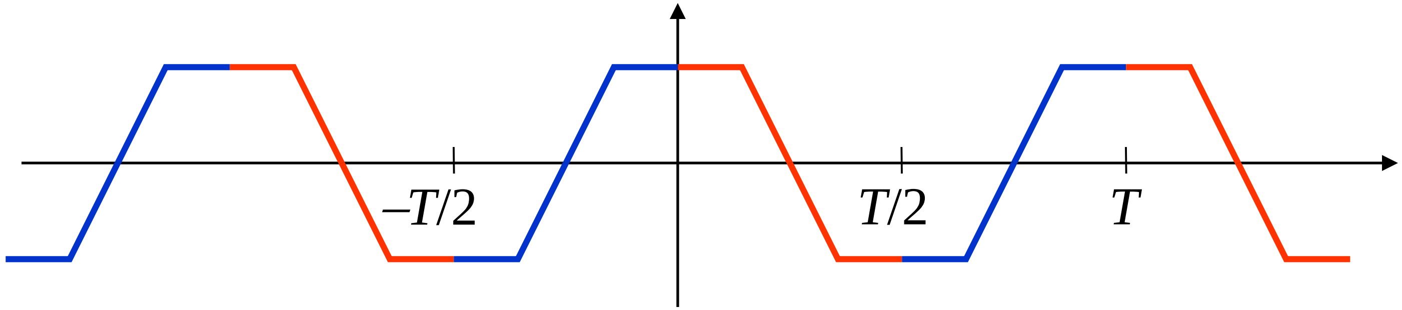
$$f(t) = f(t+T) \quad \text{and} \quad f(t) = -f\left(t + T/2\right)$$

$$\Rightarrow \quad f(t) = \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$a_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \dfrac{4}{T} \displaystyle\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt & \text{for } n \text{ odd} \end{cases}$$

$$b_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \dfrac{4}{T} \displaystyle\int_0^{T/2} f(t)\sin(n\omega_0 t)\,dt & \text{for } n \text{ odd} \end{cases}$$

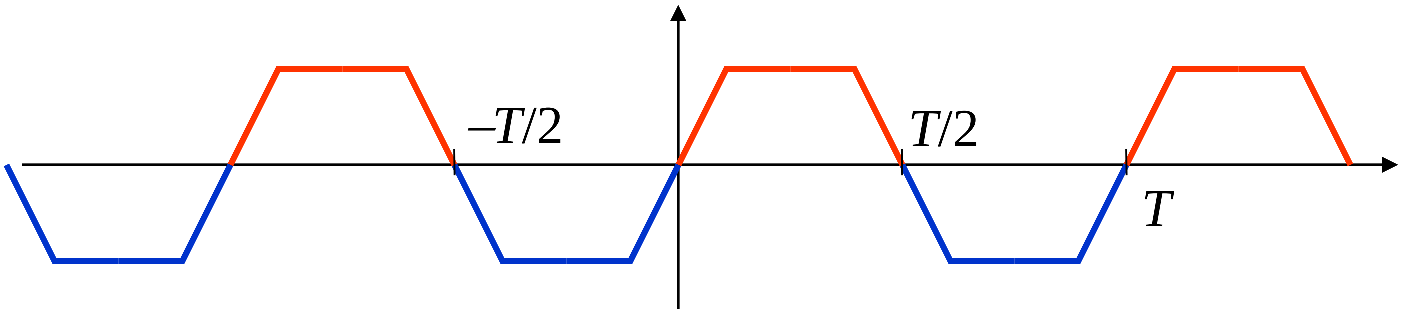
Fourier Coefficients for Even Quarter-Wave Symmetry



$$f(t) = \sum_{n=1}^{\infty} a_{2n-1} \cos[(2n-1)\omega_0 t]$$

$$a_{2n-1} = \frac{8}{T}\int_0^{T/4} f(t)\cos[(2n-1)\omega_0 t]dt$$

Fourier Coefficients for Odd Quarter-Wave Symmetry

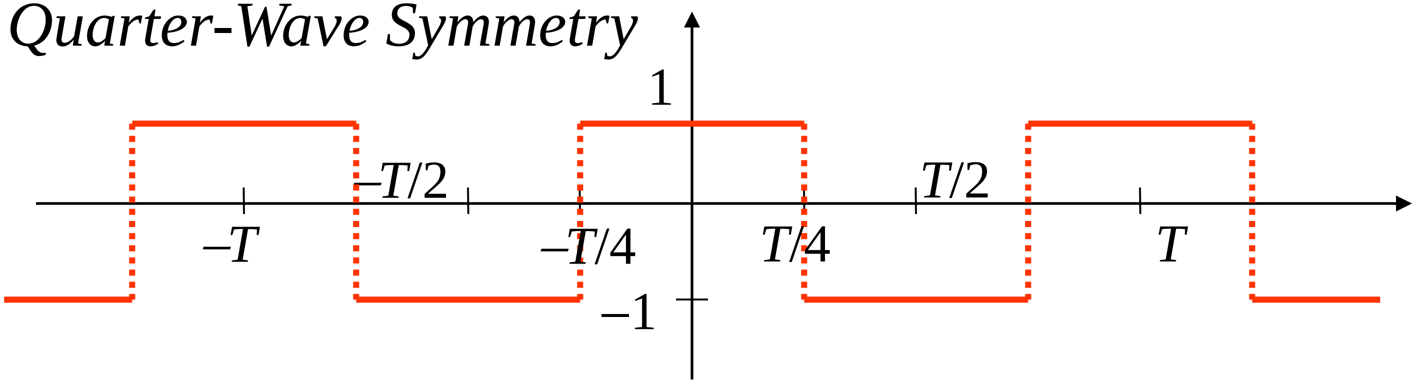


$$f(t) = \sum_{n=1}^{\infty} b_{2n-1} \sin[(2n-1)\omega_0 t]$$

$$b_{2n-1} = \frac{8}{T}\int_0^{T/4} f(t)\sin[(2n-1)\omega_0 t]dt$$

Example

Even Quarter-Wave Symmetry



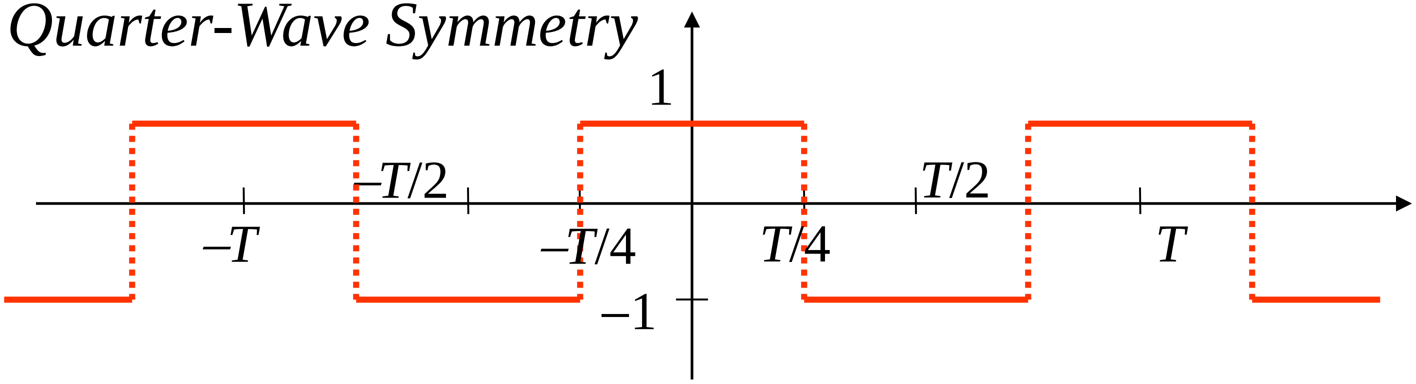
$$a_{2n-1} = \frac{8}{T}\int_0^{T/4} f(t)\cos[(2n-1)\omega_0 t]dt = \frac{8}{T}\int_0^{T/4} \cos[(2n-1)\omega_0 t]dt$$

$$= \frac{8}{(2n-1)\omega_0 T}\sin[(2n-1)\omega_0 t]\Big|_0^{T/4} = (-1)^{n-1}\frac{4}{(2n-1)\pi}$$

$$f(t)=\frac{4}{\pi}\left(\cos\omega_0 t-\frac{1}{3}\cos 3\omega_0 t+\frac{1}{5}\cos 5\omega_0 t+\cdots\right)$$

Example

Even Quarter-Wave Symmetry

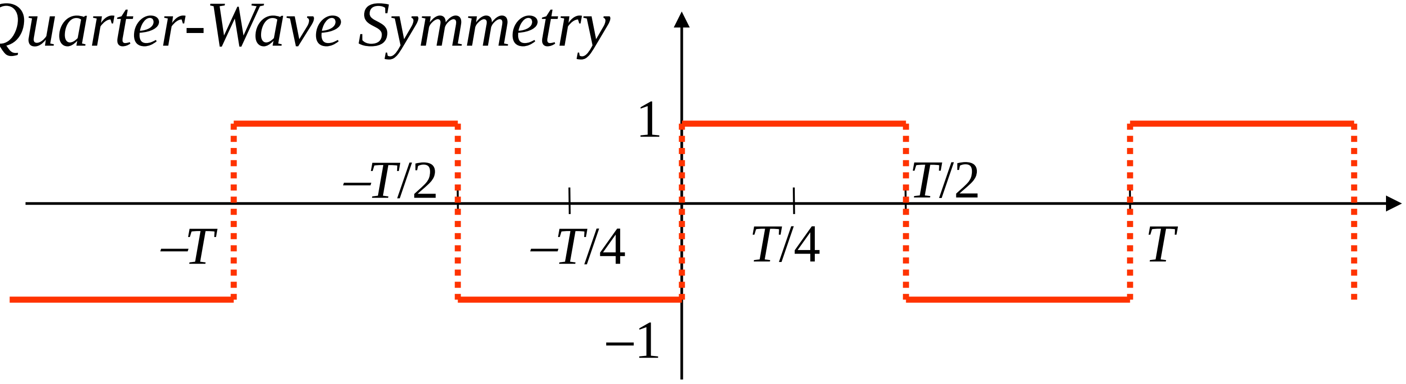


$$a_{2n-1}=\frac{8}{T}\int_0^{T/4} f(t)\cos[(2n-1)\omega_0 t]dt=\frac{8}{T}\int_0^{T/4}\cos[(2n-1)\omega_0 t]dt$$

$$=\frac{8}{(2n-1)\omega_0 T}\sin[(2n-1)\omega_0 t]\Big|_0^{T/4}=(-1)^{n-1}\frac{4}{(2n-1)\pi}$$

Example

Odd Quarter-Wave Symmetry



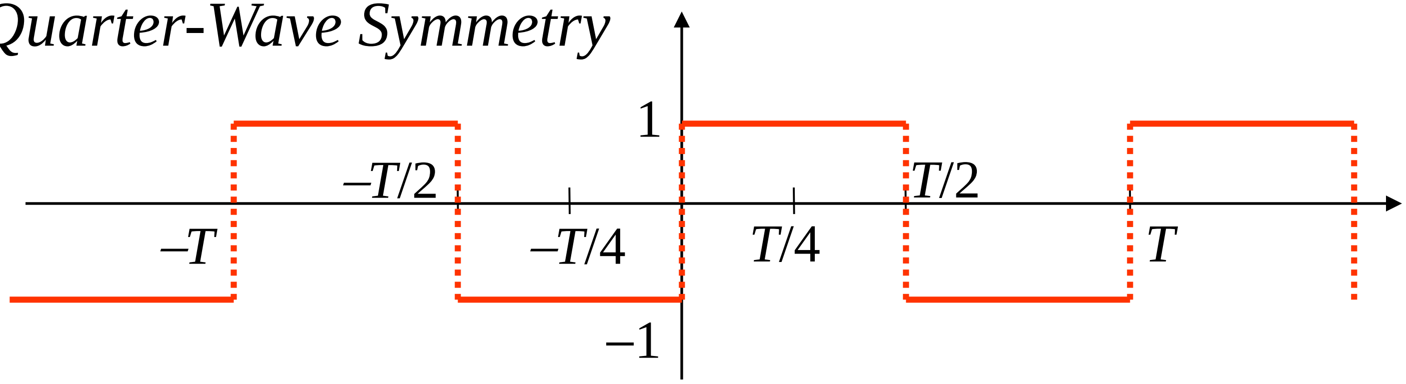
$$b_{2n-1} = \frac{8}{T}\int_0^{T/4} f(t)\sin[(2n-1)\omega_0 t]dt = \frac{8}{T}\int_0^{T/4} \sin[(2n-1)\omega_0 t]dt$$

$$= \frac{-8}{(2n-1)\omega_0 T}\cos[(2n-1)\omega_0 t]\Big|_0^{T/4} = \frac{4}{(2n-1)\pi}$$

$$f(t)=\frac{4}{\pi}\left(\sin\omega_0 t+\frac{1}{3}\sin 3\omega_0 t+\frac{1}{5}\sin 5\omega_0 t+\cdots\right)$$

Example

Odd Quarter-Wave Symmetry



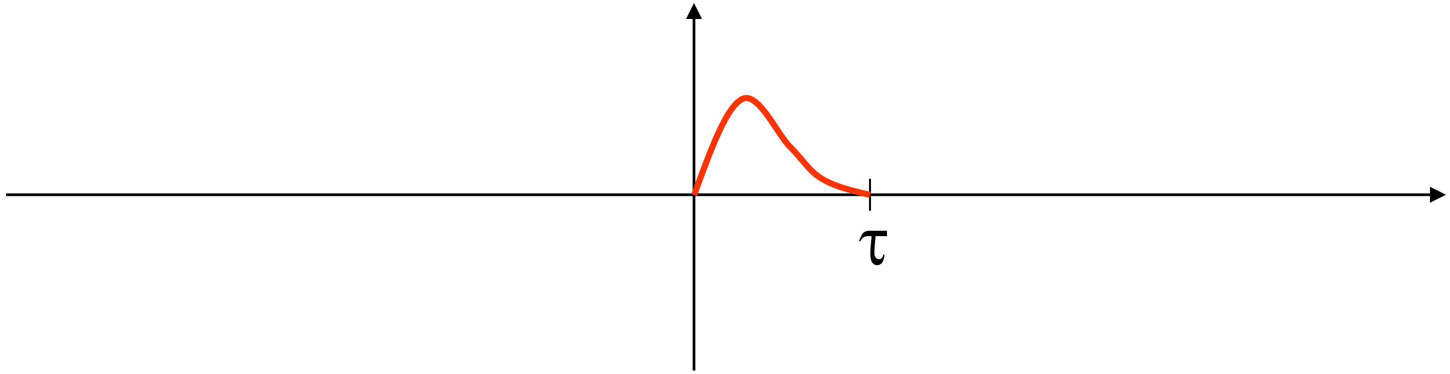
$$b_{2n-1}=\frac{8}{T}\int_0^{T/4} f(t)\sin[(2n-1)\omega_0 t]dt=\frac{8}{T}\int_0^{T/4}\sin[(2n-1)\omega_0 t]dt$$

$$=\frac{-8}{(2n-1)\omega_0 T}\cos[(2n-1)\omega_0 t]\Big|_0^{T/4}=\frac{4}{(2n-1)\pi}$$

Fourier Series

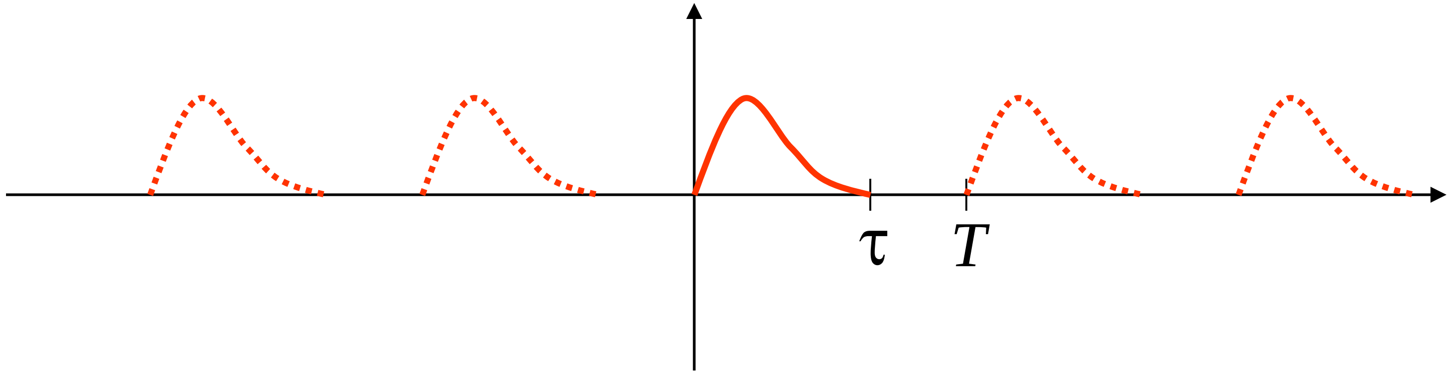
Half-Range Expansions

Non-Periodic Function Representation



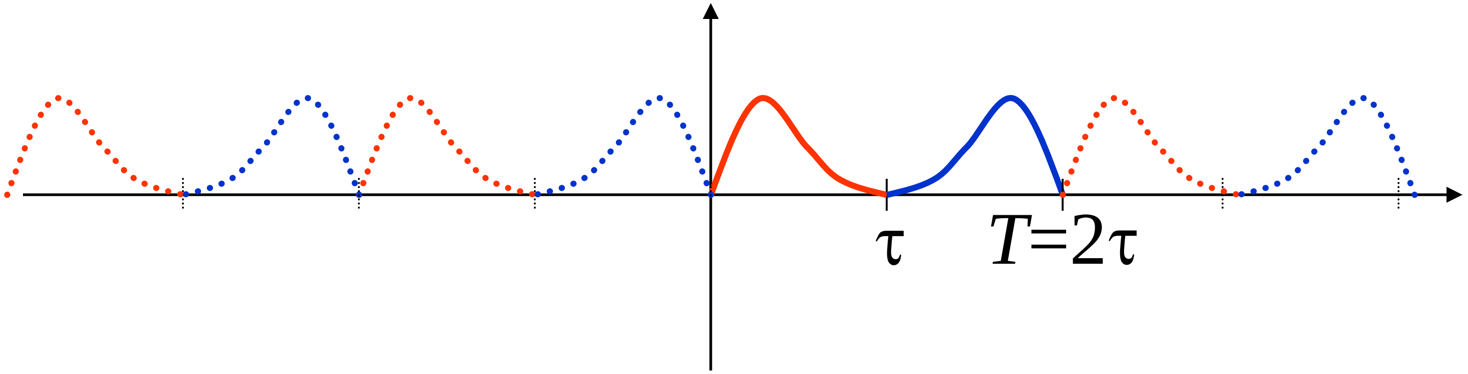
- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Without Considering Symmetry



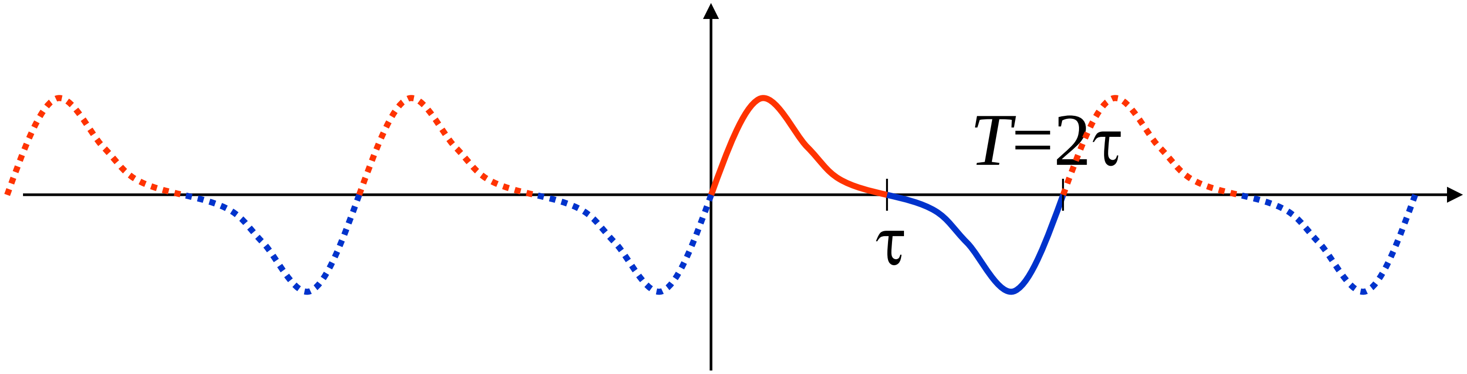
- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Even Symmetry



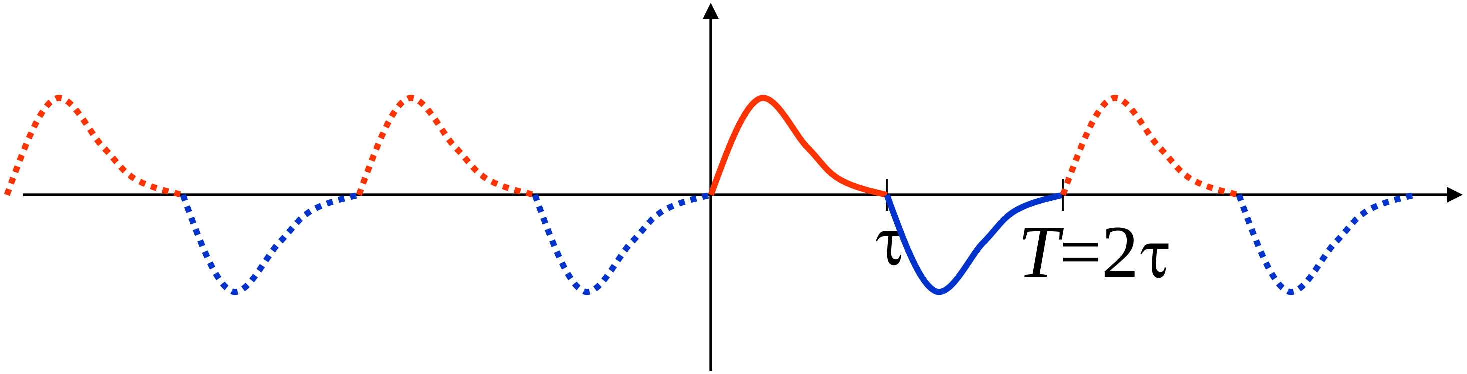
- A non-periodic function $f(t)$ defined over (0, τ) can be expanded into a Fourier series which is defined only in the interval (0, τ).

Expansion Into Odd Symmetry



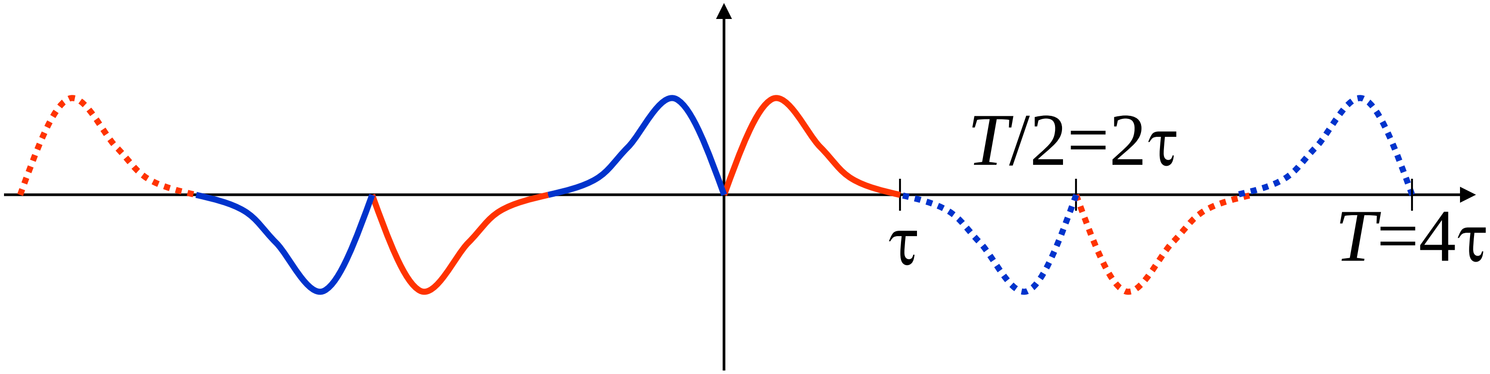
- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Half-Wave Symmetry



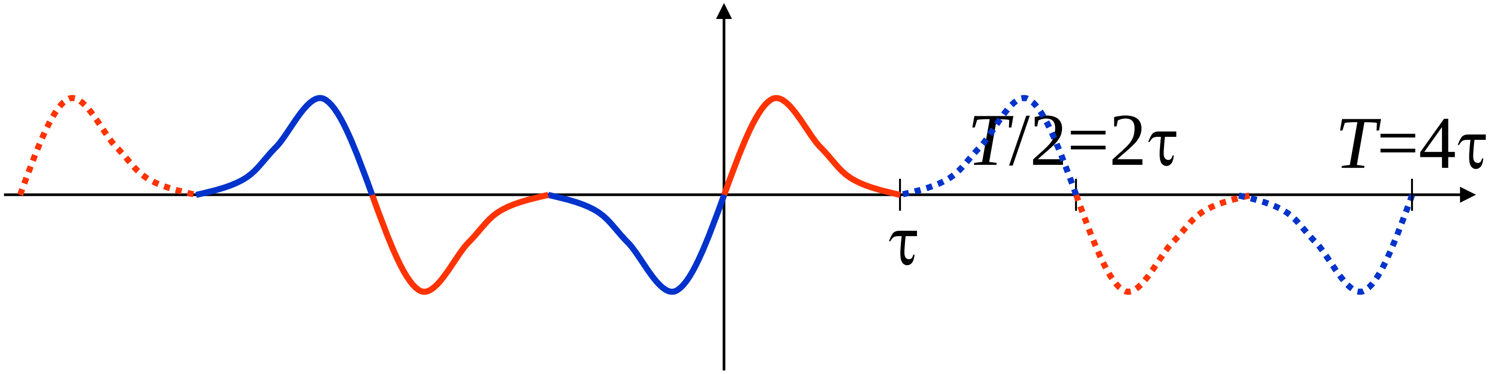
- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Even Quarter-Wave Symmetry



- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Expansion Into Odd Quarter-Wave Symmetry



- A non-periodic function $f(t)$ defined over $(0, \tau)$ can be expanded into a Fourier series which is defined only in the interval $(0, \tau)$.

Fourier Series

Least Mean-Square Error Approximation

Approximation a function

Use $S_N(t) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right)$

to represent $f(t)$ on interval $-T/2 < t < T/2$.

Define $e_N(t) = f(t) - S_N(t)$

$$\overline{\varepsilon}_N^2 = \frac{1}{T} \int_{-T/2}^{T/2} [e_N(t)]^2 \, dt$$

Mean-Square Error

Approximation a function

Show that using $S_N(t)$ to represent $f(t)$ has least mean-square property

$$\bar{\varepsilon}_N^2 = \frac{1}{T}\int_{-T/2}^{T/2} [e_N(t)]^2 dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2}\left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{N}\left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right)\right]^2 dt$$

Proven by setting $\partial\bar{\varepsilon}_N^2 / \partial a_i = 0$ and $\partial \bar{\varepsilon}_N^2 / \partial b_i = 0$.

Approximation a function

$$\overline{\varepsilon}_N^2 = \frac{1}{T}\int_{-T/2}^{T/2}[e_N(t)]^2\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2}\left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{N}\left(a_n\cos n\omega_0 t + b_n \sin n\omega_0 t\right)\right]^2 dt$$

$$\frac{\partial\overline{\varepsilon}_N^2}{\partial a_0} = \frac{a_0}{2} - \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt = 0$$

$$\frac{\partial\overline{\varepsilon}_N^2}{\partial a_n} = a_n - \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\,dt = 0$$

$$\frac{\partial\overline{\varepsilon}_N^2}{\partial b_n} = b_n - \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t\,dt = 0$$

Mean-Square Error

$$\overline{\varepsilon}_N^2 = \frac{1}{T}\int_{-T/2}^{T/2}[e_N(t)]^2 dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2}\left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{N}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\right]^2 dt$$

$$\overline{\varepsilon}_N^2 = \frac{1}{T}\int_{-T/2}^{T/2}[f(t)]^2 dt - \frac{a_0^2}{4} - \frac{1}{2}\sum_{n=1}^{N}(a_n^2 + b_n^2)$$

Mean-Square Error

$$\overline{\varepsilon}_N^2 = \frac{1}{T}\int_{-T/2}^{T/2}[e_N(t)]^2 dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2}\left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{N}\left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right)\right]^2 dt$$

$$\frac{1}{T}\int_{-T/2}^{T/2}[f(t)]^2 dt \geq \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{N}(a_n^2 + b_n^2)$$

Mean-Square Error

$$\overline{\varepsilon}_N^2 = \frac{1}{T}\int_{-T/2}^{T/2}[e_N(t)]^2\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2}\left[f(t) - \frac{a_0}{2} - \sum_{n=1}^{N}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\right]^2 dt$$

$$\frac{1}{T}\int_{-T/2}^{T/2}[f(t)]^2\,dt = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\lim_{N\to\infty}\overline{\varepsilon}_N^2 = \lim_{N\to\infty}\left\|e_N(t)\right\|^2 = 0$$